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GK-DIMENSIONS OF CORNERS AND IDEALS

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ABSTRACT
If T is a corner in a ring R (i.e. $TRT \subseteq T$) then $\mathrm{GK}(T) = \mathrm{GK}(RT^2R)$, and hence $\mathrm{GK}(ReR) = \mathrm{GK}(eRe)$ for idempotents e. Examples are given to show that $\mathrm{GK}(eRe)$ can be any integer $< \mathrm{GK}(R)$ even for prime rings. Positive results are obtained if $\mathrm{GK}(R) \leqq 2$, and R is Goldie.

1. Introduction

This paper originated in an attempt to prove that, for every prime algebra R and an idempotent $e \in R$, then $\mathrm{GK}(R) = \mathrm{GK}(eRe)$, where $\mathrm{GK}(*)$ denotes the Gelfand–Kirillov dimension. This result is not valid in the general case as the examples in section 4 show, where we produce rings R_n of $\mathrm{GK}(R_n) = n$ with idempotents e_j such that $\mathrm{GK}(e_jR_ne_j) = j$, $0 \leqq j \leqq n$. The problem may have a positive answer if R is a prime Goldie ring, and in fact this is known for PI-rings ([7], lemma 2) and Noetherian rings [6]. In Section 3, we prove that this is valid if $\mathrm{GK}(R) \leqq 2$. Moreover, it seems that the additional restriction 'to be affine' is not sufficient as shown in the last example (4.4). The subring eRe is generalized to the idea of a 'corner' $T \subseteq R$, which is a subalgebra (not necessarily with a unit) such that $TRT \subseteq T$. For these subrings we prove that $\mathrm{GK}(T) = \mathrm{GK}(RT^2R)$. This leads to some interesting corollaries (3.2). It also shows that there is a close relation between the problem stated above and the question whether $\mathrm{GK}(R) = \mathrm{GK}(A)$ for every non-zero ideal A in the prime ring R, and, in fact, similar answers exit for this problem.

All rings considered in this paper are algebras over a field k and generally with a unit unless stated otherwise.

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2. GK-dim of corners and ideals

We begin with the notion of a corner of a ring.

2.1. DEFINITION. A *corner* T of a k-algebra R is a subalgebra T which satisfies $TRT \subseteq T$.

'Corners' have also been known as 'subideals,' but we prefer this name which recalls the notion of a corner of a matrix ring. Basic examples of corners are $T = L \cap I$, the intersection of a left ideal L and a right ideal I. Examples of such are the subrings eRe, where $e^2 = e$ is an idempotent in R. Another example of two corners R, S of a ring are the two rings of a Morita context (R, U, V, S) (e.g. [1]), where both rings are corners of the ring

$$\mathfrak{R} = \begin{pmatrix} R & U \\ V & S \end{pmatrix}.$$

In fact, every corner T, or even $k + T = T^*$, the ring obtained by adding a unit to T, together with R are both corners, by "participating" in the Morita context (R, RT, TR, T^*) where the bilinear forms $RT \times TR \to R$ and $TR \times RT \to T$ are induced by multiplication in R.

REMARK. We shall refer to the GK-dim of k-algebra A, even if A does not contain a unit by setting $\mathrm{GK}(A) = \mathrm{GK}(k + A)$ where $k + A$ is the ring obtained by adjoining a unit to A. This means that if V is a finitely generated k-subspace of A, and

$$U^n = \sum_{\nu=1}^{n} V^\nu$$

then

$$GK(A) := \limsup_V (\log \dim U^n / \log n).$$

Our main result in this section is the relation between GK-dim of corners and of two-sided ideals.

2.2. THEOREM. *If T is a corner of R then* $\mathrm{GK}(T) = \mathrm{GK}(RT^2R)$.

PROOF. Let V be a k-subspace of RT^2R of finite dimension. Thus, V is a subspace of a k-space U generated by elements of the form $u_{ij} = r_i t_i t_j r_j$, $r_i \in R$, $t_i \in T$, and by adding the unit to these subspaces we obtain $V^n \subseteq U^n$ for $n \geqq 1$. Now U^n is spanned by products

$$q = (r_{i_1}t_{i_1}t_{j_1}r_{j_1})(r_{i_2}t_{i_2}t_{j_2}r_{j_2})\cdots(r_{i_m}t_{i_m}t_{j_m}r_{j_m}),$$

so writing $w_{ij} = t_ir_ir_jt_j$, then

$$q = r_{i_1}t_{i_1}(w_{j_1i_2}w_{j_2i_3}\cdots w_{j_{m-1}i_m})t_{j_m}r_{j_m}.$$

Hence, if W is the k-space spanned by $\{1, w_{ij}\}$, then $W \subseteq T$ since T is a corner of R and $q \in \Sigma_{i,j} r_it_iW^{m-1}t_jr_j$; therefore, $V^n \subseteq \Sigma\, r_it_iW^nt_jr_j$. Thus, $\dim V^n \leqq t^2 \dim W^n$ where t is the number of generators of U.

Now choose V and n large enough so that $\log\dim V^n/\log n \geqq \rho - \varepsilon$ for a fixed $\varepsilon > 0$, where $\rho = \mathrm{GK}(RT^2R)$; and noting that $W \subseteq T$ implies that, for large n, $\log\dim W^n/\log n < \sigma + \varepsilon$ where $\sigma = \mathrm{GK}(T)$, it follows readily that $\rho - \varepsilon \leqq \mathrm{GK}(T)$.

Next, $\mathrm{GK}(RT^2R) \geqq \mathrm{GK}(T^2)$ since $RT^2R \supseteq T^2$. So the last step in the proof is to show that $\mathrm{GK}(T^2) = \mathrm{GK}(T)$, which is required because T might not contain a unit.

Indeed, let

$$V = \sum_{i=1}^{s} kt_i$$

be a finite dimensional k-subspace of T generated by t_i and let $U = \Sigma\, kt_it_j$ the subspace of T^2 generated by the products t_it_j. Clearly $V^{2m} \subseteq U^m$ and

$$V^{2m+1} \subseteq \sum_{i=1}^{s} U^m t_i,$$

hence

$$\sum_{\nu\leqq n} V^\nu \subseteq \sum_{\nu\leqq n/2}\sum_{i=0}^{s} U^\nu t_i,$$

where we set $t_0 = 1$. Thus,

$$\dim\left(\sum_{\nu\leqq n} V^\nu\right) \leqq (s+1)\dim\left(\sum_{\nu\leqq n/2} U^\nu\right).$$

We continue by taking the log of both sides and dividing by $\log n$. Choosing V and n so that

$$\log\left(\dim\sum_{\nu\leqq n} V^\nu\right)\Big/\log n \geqq \sigma - \varepsilon,$$

where $\sigma = \mathrm{GK}(T)$ and n is large enough so that also

$$\log\left(\dim \sum_{\nu \leqq n/2} U^{\nu}\right) \Big/ \log\left[\frac{n}{2}\right] \leqq \tau + \varepsilon$$

where $\tau = \mathrm{GK}(T^2)$. Thus

$$\sigma - \varepsilon \leqq (\log(s+1) + (\tau + \varepsilon)\log[n/2])/\log n$$

and as $n \to \infty$, we get $\sigma - \varepsilon \leqq \tau + \varepsilon$ for every $\varepsilon > 0$, and hence $\mathrm{GK}(T) = \sigma \leqq \tau = \mathrm{GK}(T^2)$. The other direction of the inequality $\mathrm{GK}(T^2) \leqq \mathrm{GK}(T)$ is trivial since T is a ring and $T^2 \subseteq T$.

This theorem has many interesting corollaries, new and old.

2.2. Corollary 1. *If R is a simple k-algebra and e is an idempotent, then* $\mathrm{GK}(R) = \mathrm{GK}(eRe)$.

Indeed, $T = eRe$ is a corner of R and $RT^2R = R$. It yields also an alternative proof to:

Corollary 2. $\mathrm{GK}(M_n(S)) = \mathrm{GK}(S)$, *where $M_n(S)$ is the $n \times n$ matrix ring over S, or even the ring of all finite matrices over S.*

Here, we use $R = M_n(S)$, and the corner $T = e_{11}Re_{11} \cong S$ where e_{11} is the idempotent having one in the $(1,1)$ position and zero elsewhere. Again, $RT^2R = R$. We also have another proof of the fact:

Corollary 3. *If R and S are Morita equivalent, then* $\mathrm{GK}(R) = \mathrm{GK}(S)$.

In this case S is known ([5] p. 83) to be isomorphic to some $eM_n(R)e$, for some n, and idempotent $e \in M_n(R)$ with $M_n(R)eM_n(R) = M_n(R)$. Another way is to consider both rings as corners of the ring $\mathfrak{R}$ defined in (2.1).

3. GK-dim of ideals

This leads us to another question concerning GK-dim. Given $0 \neq A$ an ideal in R, we may have $\mathrm{GK}(A) < \mathrm{GK}(R)$ as well as $\mathrm{GK}(eRe) < \mathrm{GK}(R)$, and examples will be given in the last section. It is not known if this behavior can happen for prime Goldie rings. Our theorem leads to the fact that these two properties are equivalent for prime Goldie rings, namely:

3.1. Theorem. *If, for some prime Goldie ring R, there exists a non-zero ideal A such that* $\mathrm{GK}(A) < \mathrm{GK}(R)$, *then there exists a prime Goldie ring S with an idempotent e such that* $\mathrm{GK}(eSe) < \mathrm{GK}(S)$, *and the converse also holds.*

PROOF. We first show that the converse is an immediate consequence of the theorem in section 1. For, if S exists and $\mathrm{GK}(eSe) < \mathrm{GK}(S)$, then $\mathrm{GK}(eSe) = \mathrm{GK}(SeS) < \mathrm{GK}(S)$ and $SeS = A \neq 0$ is an ideal in S.

To prove the first part of the theorem, given R and A, we must consider a different ring S. Put

$$S := \begin{pmatrix} k+A & R \\ A & R \end{pmatrix}$$

which can be easily shown to be prime Goldie, and if

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad eSe = \begin{pmatrix} k+A & 0 \\ 0 & 0 \end{pmatrix}$$

and so $\mathrm{GK}(R) = \mathrm{GK}(S) > \mathrm{GK}(eSe)$ as required.

3.2. With the exceptions of the Noetherian [6] and PI [7] cases mentioned in the introduction, the only positive result of some generality appears to be:

THEOREM. *Let R be a prime Goldie ring and $0 \neq A$ an ideal of R*

(1) *If* $\mathrm{GK}(A) \leqq 1$, *then* $\mathrm{GK}(A) = \mathrm{GK}(R) = \mathrm{GK}(Q(R)) = 1$, *where* $Q(R)$ *is the ring of quotients of* R.

(2) *If* $\mathrm{GK}(R) \leqq 2$ *then* $\mathrm{GK}(A) = \mathrm{GK}(R)$ *for every ideal* A. *Hence, also for a corner* T *either* $T^2 = 0$ *and so* $\mathrm{GK}(T) = 0$ *or* $\mathrm{GK}(T) = \mathrm{GK}(R)$.

PROOF. We begin with the proof of (1), and observe that for a prime Goldie ring R, and $0 \neq A$ an ideal in R, then $k + A$ is also Goldie and prime and quotients $Q(R) = Q(k + A) = Q(A)$. We next consider various cases:

Case 1: $\mathrm{GK}(SA) = 0$, this means that A is locally finite ([4] p. 15), but, then, $Q(A) = A$ which implies that R is simple and so $Q(R) = R = A = Q(A)$, and proves (1) of our theorem in this case.

Case 2: $\mathrm{GK}(A) = 1$, then A is locally a PI-algebra, i.e. every finitely generated k-subalgebra of A satisfies a polynomial identity ([8]).

Consider first the subcase where R is assumed to be a domain, then every finitely generated subalgebra is a domain. Let

$$V = \sum_{i=1}^{r} kv_i$$

be a subspace of the quotient ring $Q(R)$ which is generated by r elements and let $V_1 = A$. Since $Q(R) = Q(A)$, we may assume that $v_i = q^{-1}a_i$ where $q, a_i \in A$. Moreover, because the k-subalgebra $R_0 = k[a_r, \ldots, a_r, q]$ is a PI-domain, its non-zero ideals, e.g. $A \cap R_0$, contain regular central elements. Hence, we can now set $v_i = p^{-1}u_i$, where $u_i \in A$, $p \in A$ and p centralizes all elements of R_0.

Clearly $V^n \subseteq p^{-n}U^n$, where

$$U = \sum_{i=1}^{r} ku_i + k \subseteq k + A,$$

and, thus, $\dim V^n \leqq \dim U^n$. This being true for all finitely generated k-subspaces $V \subseteq Q(R)$, will imply, by standard computations (as in the previous section), that $\mathrm{GK}(Q(R)) \leqq \mathrm{GK}(A)$. On the other hand, $\mathrm{GK}(A) \leqq \mathrm{GK}(R) \leqq \mathrm{GK}(Q(R))$, and this completes the proof of (1) of the theorem in Case 2.

Consider now the general prime Goldie ring R with the ideal $A \neq 0$. We know that $Q(A) = Q(R) = M_n(D)$, where D is a division algebra. Also, it is well known ([2]) that D is isomorphic to a quotient ring of a Goldie domain $\Delta \cong I/I \cap l(I)$, where I is a minimal annihilator in R which can be chosen to satisfy $A \cap I \not\subseteq l(I)$. So now we have $\mathrm{GK}(A \cap I/I \cap l(I)) \leqq 1$ and so $\mathrm{GK}(\Delta) \leqq 1$ by the previous cases and consequently $\mathrm{GK}(D) = \mathrm{GK}(Q(\Delta)) \leqq 1$, which implies $\mathrm{GK}(M_n(D)) \leqq 1$. Finally

$$1 \leqq \mathrm{GK}(A) \leqq \mathrm{GK}(Q(A)) = \mathrm{GK}(Q(R)) = \mathrm{GK}(M_n(D)) \leqq 1$$

and part (1) of our theorem is proved.

The second part of the theorem now follows immediately, since $\mathrm{GK}(A) \leqq \mathrm{GK}(R) \leqq 2$, and if $\mathrm{GK}(R) = 2$, we cannot have $\mathrm{GK}(A) \leqq 1$ by part (1). Also, if T is a corner of R then $\mathrm{GK}(T) = \mathrm{GK}(RT^2R)$, $RT^2R = A$ is an ideal in R. If $A = 0$, then $T^2 \neq 0$ since R is prime and so $\mathrm{GK}(T) = 0$; for $A \neq 0$, the proof follows from (1) of the theorem.

The following is an interesting reformulation of part of our theorem.

Corollary. *If R is a prime Goldie ring, and* $\mathrm{GK}(R) \geqq 2$ *then* $\mathrm{GK}(A) \geqq 2$ *for every non-zero ideal A in R.*

4. Examples

Our first example is a prime ring R with idempotents e_i such that $\mathrm{GK}(e_iRe_i) = i$ for $0 \leqq i < n$ and $\mathrm{GK}(R) = n$.

We begin with an arbitrary ring R, and the ring R_f of all square matrices over R with countably many rows, and with only finitely many non-zero entries in each row. In this ring we consider the subring R_b of all finite matrices (of bounded rows and columns); we identify R with the diagonal matrices, and consider the matrix

$$E = \sum_{i=1}^{\infty} e_{i\,i+1}.$$

4.1. THEOREM. *If R is prime, then $S = R_b + R[E]$ is prime*; $\mathrm{GK}(S) = \mathrm{GK}(R) + 1$, *and S contains an idempotent e such that $eSe \cong R$, and so* $\mathrm{GK}(eSe) = \mathrm{GK}(R) < \mathrm{GK}(S)$. *Also, S contains the ideal R_b, for which* $\mathrm{GK}(S) > \mathrm{GK}(R_b) = \mathrm{GK}(R)$.

PROOF. One quickly verifies that S is a prime ring. If V is a finitely generated k-subspace of S, it can be embedded in a k-space U generated by elements $r_j \in R$ (set $r_0 = 1$), the matrix E and a finite set of matrices $\mathscr{A}_\lambda$ of the form $\{r_{ik}e_{ik};\, i, k \leqq m\}$. Let U_1 be the k-space generated by the matrices $\mathscr{A}_\lambda$, and all $E^\nu \mathscr{A}_\lambda$ and $\{r_j e_{ii};\, e \leqq m\}$, for all $r_j \in U$; hence, $U_1 \subseteq M_m(R)$. Note also that only a finite set of $E^\nu \mathscr{A}_\lambda = 0$ since $E^{m+1}\mathscr{A}_\lambda = 0$. Let U_0 be the k-space spanned by the r_j, $r_j \in U_0$. Thus

$$V^n \subseteq \sum_{y,\mu \leqq n} U_1^\nu E^\mu + \sum_{\nu=0}^{n} U_0^n E^\nu$$

which yields $\dim V^n \leqq (n+1)\dim U_1^n + (n+1)\dim U_0^n \leqq 2(n+1)\dim U_1^n$, since U_0^n is isomorphic to a subspace of U_1^n generated by $\{\Sigma_{i=1}^m r_j e_{ii}\}$ for all j. By standard computation choosing appropriate V and letting $n \to \infty$, we get $\mathrm{GK}(S) \leqq 1 + \mathrm{GK}(M_m(R)) = 1 + \mathrm{GK}(R)$.

On the other hand, $S \supseteq R[E]$, and clearly $R[E] \cong R[x]$, the ring of polynomials in a commutative indeterminate x. Hence, $\mathrm{GK}(S) \geqq \mathrm{GK}(R[x]) = \mathrm{GK}(R) + 1$ which proves the first property of S.

Finally, let e be any finite idempotent

$$e = \sum_{i=1}^{n} e_{ii},$$

where e_{ii} is the matrix with A in the (i, i) place and zero elsewhere. Then, clearly, $eSe \cong M_n(R)$, and so $\mathrm{GK}(eSe) = \mathrm{GK}(M_n(R)) = \mathrm{GK}(R)$. The statement of our theorem is for the case $n = 1$. The rest of the theorem follows easily by noting that R_b is a two-sided ideal in S, and every finitely generated subspace of R belongs to some finite ring of matrices so that $\mathrm{GK}(R_b) = \mathrm{GK}(R)$.

Another interesting property of S is:

4.2. COROLLARY. *If R has finite (classical) Krull dim, then* K-dim$(S) \geqq 2(\text{K-dim}(R) + 1)$.

Indeed, R_b is a prime ideal in S, and $S/R_b \cong R[E] \cong R[x]$. Hence,

$$\text{K-dim}(S/R_b) = \text{K-dim}\, R[x] \geqq \text{K-dim}\, R + 1,$$

as for a commutative ring.

But also S has a chain of prime ideals $(P_i)_f \subset R_f$ where $R \supset P_1 \supset P_2 \supset \cdots \supset P_r \supset 0$, $r = \text{K-dim}(R)$ is a maximal chain of prime ideals of R. Thus, together with R_f we have a chain of primes of length $r + 1$, which proves that Krull dim $S \geqq 2(\text{K-dim}(R) + 1)$.

4.3. Denote the ring constructed above $S := F(R)$ and by induction let $R_0 := k$, $R_{n+1} := F(R_n)$, then we have:

THEOREM. $\text{GK}(R_n) = n$, $\text{K-dim}(R_n) \geqq 2^n$; *$R_n$ contains idempotents e_j, $j = 0, 1, \ldots, n-1$ such that* $\text{GK}(e_jR_ne_j) = \text{GK}(R_ne_jR_n) = j$.

The proof follows by induction. Assume that for $j = 0, \ldots, n-1$ the rings R_j satisfy the conclusion of the theorem $R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_{n-1}$, and say that for $k \leqq j \leqq n-1$ there exists idempotents $e_k^{(j)}$, $k \leqq j$ which satisfy $e_k^{(j)}R_je_k^{(j)} \cong R_k$.

Clearly, the ring $R_n = F(R_{n-1})$ contains R'_{n-1} as the diagonal matrices. Let e_n be the idempotent having one in (1, 1) place, then, letting $e_k^{(n)}$ be the inverse image of $e_k^{(n-1)}$ in $e_nR_ne_n \subset R_n$ will complete the induction process of construction. Finally $\text{GK}(e_k^{(n)}R_ne_k^{(n)}) = \text{GK}(R_k) = k$ by induction.

4.4. We note that the preceding examples are prime but not affine; yet, we can even show that a well known affine k-algebra has a similar property.

THEOREM. *Set $k[x, y]$ the free ring in two non-commutative variables x, y, and $R := k[x, y]/\{yx - 1\}$ where $\{yx - 1\}$ is the ideal generated by $yx - 1$. Then R is a prime, even primitive, ring with* Krull $\dim(R) = \text{GK}(R) = 2$, *but $e = 1 - xy$ is an idempotent in R such that* $\text{GK}(eRe) = \text{GK}(ReR) = 0$. *Moreover* $\text{GK}(R/ReR) = 1$.

The fact that R is prime and even primitive was shown in ([3]). Let $R = k[s, t]$ where $s = \bar{x}$, $t = \bar{y}$ are the classes of x and y; then $ts = 1$. Every element of R can be written uniquely in the form $f = \Sigma\, \alpha_{u_\mu} s^\nu t^\mu$ which yields that $\text{GK}(R) = 2$.

Let $e = 1 - st$; then $te = 0$, $es = 0$, and so $eRe = ke$ which forces

$GK(eRe) = 0$. It follows, from the theorem of the first section, that $GK(ReR) = 0$. Finally, $R/ReR \cong k[u, u^{-1}]$ by mapping $t \to u$, $s \to u^{-1}$, which is an isomorphism since $\bar{x}\bar{y} = \bar{y}\bar{x} = 1 \bmod ReR$ and so $GK(R/ReR) = 1$. Here we have an exact sequence $0 \to A \to R \to R/A \to 0$, with $GK(A) = 0$, $GK(R/A) = 1$ and $GK(R) = 2$, and R is prime and affine with GK-dim taken as rings.

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